

Symmetry, Integrable Chain Models and Stochastic Processes

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Abstract

A general way to construct chain models with certain Lie algebraic or quantum Lie algebraic symmetries is presented. These symmetric models give rise to series of integrable systems. As an example the chain models with A_n symmetry and the related Temperley-Lieb algebraic structures and representations are discussed. It is shown that corresponding to these A_n symmetric integrable chain models there are exactly solvable stationary discrete-time (resp. continuous-time) Markov chains whose spectra of the transition matrices (resp. intensity matrices) are the same as the ones of the corresponding integrable models.

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1 Introduction

Integrable chain models have been discussed for many years in statistical and condensed matter physics. Some of them have been obtained and investigated using an algebraic “Bethe Ansatz method” [1], see e.g., [2] for periodic boundary conditions and [3] for fixed boundary conditions. The intrinsic symmetry of these integrable chain models plays an essential role in finding complete sets of eigenstates of the systems.

On the other hand, stochastic models like stochastic reaction-diffusion models, models describing coagulation/decoagulation, birth/death processes, pair-creation and pair-annihilation of molecules on a chain, have attracted considerable interest due to their

importance in many physical, chemical and biological processes [4]. Some of these stochastic models can be “exactly solved”, see e.g., [5]. The theoretical description of stochastic reaction-diffusion systems is given by the “master equation” which describes the time evolution of the probability distribution function [6]. This equation has the form of a heat equation with potential (i.e., a Schrödinger equation with “imaginary time”). If an integrable system can be transformed into a stochastic reaction-diffusion system, e.g., by a unitary transformation between their respective Hamiltonians, looked upon as self-adjoint operators acting in the respective Hilbert spaces, then the stochastic model so obtained is exactly soluble with the same energy spectrum as the one of the integrable system [7].

In this paper, by using the coproduct properties of bi-algebras, we present a general procedure for the construction of chain models having a certain Lie algebra or quantum Lie algebra symmetry with nearest or non-nearest neighbours interactions. The models obtained in this way can be reduced to integrable ones via a detailed representation of the symmetry algebras involved. As an example we discuss integrable models with A_n symmetry. These models turn out to have an additional Temperley-Lieb (TL) algebraic structure, in the sense that the Hamiltonians give rise to unitary representations of the TL algebra and can be expressed by the elements of the TL algebra. What is more, we find all these models can be transformed into both stationary discrete-time and stationary continuous-time Markov chains (discrete reaction-diffusion models, see e.g. [8]), whose spectra of the transition matrices resp. intensity matrices are the same as the ones of these A_n invariant integrable models.

In section 2 we first recall the basic properties of bialgebras and then describe their use in the construction of chain models with a Lie algebra symmetry resp. a quantum Lie algebra symmetry. In section 3 we discuss the A_n symmetric integrable models in the fundamental representation of A_n . We give the representation of a TL algebra in these models. In section 4 we prove that these A_n symmetric integrable chain models can be transformed into both continuous-time and discrete-time Markov chains. Some conclusions and remarks are given in section 5.

2 Chain Models with Algebraic Symmetry

2.1 Bi-algebra

Let A be an associative algebra. A is said to be a bi-algebra if it contains two linear operators, the multiplication m and the coproduct Δ .

The operation of multiplication m is defined by:

$$\begin{aligned} m : A \otimes A &\rightarrow A \\ m(a \otimes b) &= ab, \quad \forall a, b \in A. \end{aligned} \tag{1}$$

Let $\{e_i\}$ be the set of base elements of the algebra A . Then (1) means

$$m(e_i \otimes e_j) = \sum_k m_{ij}^k e_k. \tag{2}$$

The tensor m_{ij}^k determines the properties of m completely.

The multiplication m is associative,

$$m(m \otimes \mathbf{id}) = m(\mathbf{id} \otimes m), \tag{3}$$

i.e.,

$$\sum_k m_{ij}^k m_{kl}^n = \sum_k m_{ik}^n m_{jl}^k, \tag{4}$$

where \mathbf{id} denotes the identity transformation,

$$\mathbf{id} : A \rightarrow A, \quad \mathbf{id}(a) = a.$$

However in general m is not commutative, i.e., we have

$$m \circ p \neq m, \tag{5}$$

or equivalently, $m_{ij}^k \neq m_{ji}^k$, where p is the transposition operator,

$$p : A \otimes A \rightarrow A \otimes A, \quad p(a \otimes b) = (b \otimes a), \quad \forall a, b \in A. \tag{6}$$

The coproduct operator Δ maps A into $A \otimes A$:

$$\begin{aligned} \Delta : A &\rightarrow A \otimes A, \\ \Delta(e_i) &= \sum_{jk} \mu_i^{jk} e_j \otimes e_k. \end{aligned} \tag{7}$$

The properties of the coproduct are described by the tensor μ_i^{jk} .

Δ is an algebraic homomorphism,

$$\begin{aligned}\Delta(ab) &= \Delta(a)\Delta(b), \quad \forall a, b \in A, \\ \sum_k \mu_k^{rs} m_{ij}^k &= \sum_{ktpq} m_{kp}^r m_{tq}^s \mu_i^{kt} \mu_j^{pq},\end{aligned}\tag{8}$$

where $\Delta(a)$ and $\Delta(b)$ belong to $A \otimes A$ and the multiplication of tensors is defined by

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2, \quad a_1, a_2, b_1, b_2 \in A.$$

The coproduct is associative

$$\begin{aligned}(\Delta \otimes \text{id})\Delta &= (\text{id} \otimes \Delta)\Delta, \\ \sum_t \mu_t^{rs} \mu_i^{tp} &= \sum_t \mu_t^{sp} \mu_i^{rt},\end{aligned}\tag{9}$$

but in general not co-commutative

$$p \circ \Delta \neq \Delta, \quad \mu_j^{rs} \neq \mu_j^{sr}.\tag{10}$$

The operation Δ preserves all the algebraic relations of the algebra A . It gives a way to find representations of the algebra A in the direct product of spaces.

If a bi-algebra has in addition unit, counit and antipode operators, then it is called a Hopf algebra. Lie algebras are Hopf algebras with Δ co-commutative. Quantum algebras are Hopf algebras that are not co-commutative, see e.g. [9] and references therein.

2.2 Chain Models with Lie algebraic Symmetry

Let A be a Lie algebra with basis $e = \{e_\alpha\}$, $\alpha = 1, 2, \dots, n$, satisfying the Lie commutation relations

$$[e_\alpha, e_\beta] = C_{\alpha\beta}^\gamma e_\gamma,\tag{11}$$

where $C_{\alpha\beta}^\gamma$ are the structure constants with respect to the base e .

Let Δ (resp. $C(e)$) be the coproduct operator (resp. Casimir operator) of the algebra A . We have

$$[C(e), e_\alpha] = 0, \quad \alpha = 1, 2, \dots, n.\tag{12}$$

The coproduct operator action on the Lie algebra elements is given by

$$\Delta e_\alpha = e_\alpha \otimes \mathbf{1} + \mathbf{1} \otimes e_\alpha,\tag{13}$$

$\mathbf{1}$ stands for the identity operator. It is easy to check that

$$[\Delta e_\alpha, \Delta e_\beta] = C_{\alpha\beta}^\gamma \Delta e_\gamma.$$

From the properties of the coproduct, $\Delta C(e)$ is a rank two tensor satisfying

$$[\Delta C(e), \Delta e_\alpha] = 0, \quad \alpha = 1, 2, \dots, n. \quad (14)$$

Let \mathcal{F} denote an entire function defined on the $(L+1)$ -th tensor space $A \otimes A \otimes \dots \otimes A$ of the algebra A . From (14) we have

$$[\mathcal{F}(\Delta C(e)), \Delta e_\alpha] = 0, \quad \alpha = 1, 2, \dots, n. \quad (15)$$

We consider “a chain with $L+1$ sites”, i.e., the set $\{1, 2, \dots, L+1\}$. We call it an algebraic chain in the following. To each point i of the chain we associate a (finite dimensional complex) Hilbert space H_i . We can then associate to the whole chain the tensor product $H_1 \otimes H_2 \otimes \dots \otimes H_{L+1}$. For simplicity we use subindices $i, j, k \dots$ for the points in the chain sites.

The generators of the algebra A acting on the Hilbert space $H_1 \otimes H_2 \otimes \dots \otimes H_{L+1}$ associated with the above chain are given by

$$E_\alpha = \Delta^L e_\alpha, \quad \alpha = 1, 2, \dots, n, \quad (16)$$

where we have defined

$$\Delta^m = \underbrace{(\mathbf{id} \otimes \dots \otimes \mathbf{id})}_{m \text{ times}} \otimes \Delta \dots (\mathbf{id} \otimes \mathbf{id} \otimes \Delta) (\mathbf{id} \otimes \Delta) \Delta, \quad \forall m \in \mathbb{N}. \quad (17)$$

E_α also generates the Lie algebra A ,

$$[E_\alpha, E_\beta] = C_{\alpha\beta}^\gamma E_\gamma.$$

We call

$$H = \sum_{i=1}^L \mathcal{F}(\Delta C(e))_{i,i+1} \quad (18)$$

the (quantum mechanics) Hamiltonian associated with the chain $\{1, 2, \dots, L+1\}$ and given by real entire function \mathcal{F} . Here $\mathcal{F}(\Delta C(e))_{i,i+1}$ means that the rank-two tensor element $\mathcal{F}(\Delta C(e))$ is on the sites i and $i+1$ of the chain, i.e.,

$$\mathcal{F}(\Delta C(e))_{i,i+1} = \mathbf{1}_1 \otimes \dots \otimes \mathbf{1}_{i-1} \otimes \mathcal{F}(\Delta C(e)) \otimes \mathbf{1}_{i+2} \otimes \dots \otimes \mathbf{1}_{L+1}.$$

[Theorem 1]. The Hamiltonian H is a self-adjoint operator acting in $H_1 \otimes H_2 \otimes \dots \otimes H_{L+1}$ and is invariant under the algebra A .

[Proof]. That H is self-adjoint is immediate from the definition. To prove the invariance of H it suffices to prove $[H, E_\alpha] = 0$, $\alpha = 1, 2, \dots, n$. From formula (13) E_α in (16) is simply

$$E_\alpha = \sum_{i=1}^L (e_\alpha)_i, \quad (19)$$

where

$$(e_\alpha)_i = \mathbf{1}_1 \otimes \dots \otimes \mathbf{1}_{i-1} \otimes (e_\alpha) \otimes \mathbf{1}_{i+1} \otimes \dots \otimes \mathbf{1}_{L+1}.$$

Obviously

$$[\mathcal{F}(\Delta C(e))_{i,i+1}, (e_\alpha)_j] = 0, \quad \forall j \neq i, i+1. \quad (20)$$

By using formula (20) and (13) we have

$$\begin{aligned} [H, E_\alpha] &= \left[\sum_{i=1}^L \mathcal{F}(\Delta C(e))_{i,i+1}, \sum_{j=1}^L (e_\alpha)_j \right] \\ &= \sum_{i=1}^L \left[\mathcal{F}(\Delta C(e))_{i,i+1}, \sum_{j=1}^{i-1} (e_\alpha)_j + \sum_{j=i+2}^L (e_\alpha)_j + (e_\alpha)_i + (e_\alpha)_{i+1} \right] \\ &= \sum_{i=1}^L [\mathcal{F}(\Delta C(e))_{i,i+1}, (e_\alpha)_i + (e_\alpha)_{i+1}] \\ &= \sum_{i=1}^L [\mathcal{F}(\Delta C(e))_{i,i+1}, (\Delta e_\alpha)_{i,i+1}]. \end{aligned}$$

By formula (15) we get $[H, E_\alpha] = 0$, $\alpha = 1, 2, \dots, n$. ■

The Hamiltonian system given by (18) describes nearest neighbours interactions. Systems with A -invariant Hamiltonians and non-nearest neighbours interactions can be constructed by iterating the application of the coproduct operator to the Casimir operator. For instance, the following Hamiltonian describes a system with $N+1$ ($N \leq L$) sites interactions:

$$H^N = \sum_{i=1}^{L-N+1} \mathcal{F}(\Delta^N C(e))_{i,i+1,\dots,i+N}, \quad (21)$$

with Δ^N as in definition (17). H^N commutes with the generators E_α , $\alpha = 1, 2, \dots, n$, of

the algebra A on the Hilbert space $H_1 \otimes \dots \otimes H_{L+1}$ since

$$\begin{aligned}
[H^N, E_\alpha] &= \left[\sum_{i=1}^{L-N+1} \mathcal{F}(\Delta^N C(e))_{i,i+1,\dots,i+N}, \sum_{j=1}^L (e_\alpha)_j \right] \\
&= \sum_{i=1}^{L-N+1} \left[\mathcal{F}(\Delta^N C(e))_{i,i+1,\dots,i+N}, \sum_{j=1}^{i-1} (e_\alpha)_j + \sum_{j=i}^{i+N} (e_\alpha)_j + \sum_{j=i+N+1}^L (e_\alpha)_j \right] \\
&= \sum_{i=1}^{L-N+1} \left[\mathcal{F}(\Delta^N C(e))_{i,i+1,\dots,i+N}, \sum_{j=i}^{i+N} (e_\alpha)_j \right] \\
&= \sum_{i=1}^{L-N+1} [\mathcal{F}(\Delta^N C(e)), (\Delta^N e_\alpha)]_{i,i+1,\dots,i+N} = 0,
\end{aligned}$$

where the relation $\Delta^N([\mathcal{F}(C(e)), e_\alpha]) = [\Delta^N(\mathcal{F}(C(e))), \Delta^N(e_\alpha)] = 0$ has been used.

2.3 Chain Models with Quantum Lie Algebraic Symmetry

Let $e = \{e_\alpha, f_\alpha, h_\alpha\}$, $\alpha = 1, 2, \dots, n$, be the Chevalley basis of a Lie algebra A with rank n . Let $e' = \{e'_\alpha, f'_\alpha, h'_\alpha\}$, $\alpha = 1, 2, \dots, n$, be the corresponding elements of the quantum (q-deformed) Lie algebra A_q . We denote by r_α the simple roots of the Lie algebra A . The Cartan matrix $(a_{\alpha\beta})$ is then

$$a_{\alpha\beta} = \frac{1}{d_\alpha}(r_\alpha \cdot r_\beta), \quad d_\alpha = \frac{1}{2}(r_\alpha \cdot r_\alpha). \quad (22)$$

We introduce a complex quantum parameter q , such that $q^{d_\alpha} \neq \pm 1, 0$. The quantum algebra generated by $\{e'_\alpha, f'_\alpha, h'_\alpha\}$ is defined by the following relations:

$$\begin{aligned}
[h'_\alpha, h'_\beta] &= 0, \\
[h'_\alpha, e'_\beta] &= a_{\alpha\beta} e'_\beta, \\
[h'_\alpha, f'_\beta] &= -a_{\alpha\beta} f'_\beta, \\
[e'_\alpha, f'_\beta] &= \delta_{\alpha,\beta} \frac{q^{d_\alpha h'_\alpha} - q^{-d_\alpha h'_\alpha}}{q^{d_\alpha} - q^{-d_\alpha}}
\end{aligned} \quad (23)$$

together with the quantum Serre relations

$$\begin{aligned}
\sum_{\gamma=0}^{1-a_{\alpha\beta}} (-1)^\gamma \begin{bmatrix} 1-a_{\alpha\beta} \\ \gamma \end{bmatrix}_{q^{d_\alpha}} (e'_\alpha)^\gamma e'_\beta (e'_\alpha)^{1-a_{\alpha\beta}-\gamma} &= 0, \quad i \neq j, \\
\sum_{\gamma=0}^{1-a_{\alpha\beta}} (-1)^\gamma \begin{bmatrix} 1-a_{\alpha\beta} \\ \gamma \end{bmatrix}_{q^{d_\alpha}} (f'_\alpha)^\gamma f'_\beta (f'_\alpha)^{1-a_{\alpha\beta}-\gamma} &= 0, \quad i \neq j,
\end{aligned} \quad (24)$$

where for $m \geq n \in \mathbb{N}$,

$$\begin{aligned} \begin{bmatrix} m \\ n \end{bmatrix}_q &= \frac{[m]_q!}{[n]_q![m-n]_q!}, \\ [n]_q! &= [n]_q[n-1]_q \cdots [2]_q[1]_q, \\ [n]_q &= \frac{q^n - q^{-n}}{q - q^{-1}}. \end{aligned}$$

The coproduct operator Δ' of the quantum algebra A_q is given by

$$\Delta' h'_\alpha = h'_\alpha \otimes \mathbf{1} + \mathbf{1} \otimes h'_\alpha, \quad (25)$$

$$\Delta' e'_\alpha = e'_\alpha \otimes q^{-d_\alpha h'_\alpha} + q^{d_\alpha h'_\alpha} \otimes e'_\alpha, \quad (26)$$

$$\Delta' f'_\alpha = f'_\alpha \otimes q^{-d_\alpha h'_\alpha} + q^{d_\alpha h'_\alpha} \otimes f'_\alpha. \quad (27)$$

It is straightforward to check that Δ' preserves all the algebraic relations in (23) and (24).

Let $C_q(e')$ be the Casimir operator of A_q , i.e., $[C_q(e'), a] = 0, \forall a \in A_q$. For any entire function \mathcal{F} of $C_q(e')$, we have

$$[\mathcal{F}(C_q(e')), a] = 0, \forall a \in A_q \quad (28)$$

and

$$[\Delta' \mathcal{F}(C_q(e')), \Delta' a] = 0, \forall a \in A_q. \quad (29)$$

Especially, by formula (25) one gets

$$\Delta' q^{\pm d_\alpha h'_\alpha} = q^{\pm d_\alpha h'_\alpha} \otimes q^{\pm d_\alpha h'_\alpha}. \quad (30)$$

Hence

$$[\Delta' \mathcal{F}(C_q(e')), \Delta' q^{\pm d_\alpha h'_\alpha}] = [\Delta' \mathcal{F}(C_q(e')), q^{\pm d_\alpha h'_\alpha} \otimes q^{\pm d_\alpha h'_\alpha}] = 0. \quad (31)$$

The generators of A_q on a chain with $(L + 1)$ -sites are given by

$$\begin{aligned}
H'_\alpha &= \Delta'^L h'_\alpha = \sum_{i=1}^{L+1} (h'_\alpha)_i \\
E'_\alpha &= \Delta'^L e'_\alpha, \\
&= \sum_{i=1}^{L+1} (q^{d_\alpha h'_\alpha})_1 \otimes \dots \otimes (q^{d_\alpha h'_\alpha})_{i-1} \otimes (e'_\alpha)_i \otimes (q^{-d_\alpha h'_\alpha})_{i+1} \otimes \dots \otimes (q^{-d_\alpha h'_\alpha})_{L+1}, \quad (32) \\
F'_\alpha &= \Delta'^L f'_\alpha \\
&= \sum_{i=1}^{L+1} (q^{d_\alpha h'_\alpha})_1 \otimes \dots \otimes (q^{d_\alpha h'_\alpha})_{i-1} \otimes (f'_\alpha)_i \otimes (q^{-d_\alpha h'_\alpha})_{i+1} \otimes \dots \otimes (q^{-d_\alpha h'_\alpha})_{L+1}.
\end{aligned}$$

[Theorem 2]. The chain model defined by the following Hamiltonian acting in $H_1 \otimes \dots \otimes H_{L+1}$ is invariant under the quantum algebra A_q :

$$H_q = \sum_{i=1}^L (\Delta' \mathcal{F}(C_q(e'))_{i,i+1}). \quad (33)$$

[Proof]. Using (32) and (25) we get

$$\begin{aligned}
[H_q, H'_\alpha] &= \sum_{i=1}^L \left[(\Delta' \mathcal{F}(C_q(e'))_{i,i+1}, \sum_{j=1}^{L+1} (h'_\alpha)_j \right] \\
&= \sum_{i=1}^L [(\Delta' \mathcal{F}(C_q(e'))_{i,i+1}, (h'_\alpha)_i + (h'_\alpha)_{i+1}] \\
&= \sum_{i=1}^L [(\Delta' \mathcal{F}(C_q(e'))_{i,i+1}, \Delta'(h'_\alpha))_{i,i+1}] = 0.
\end{aligned}$$

From formulae (32) and (31) we have

$$\begin{aligned}
[H_q, E'_\alpha] &= \left[\sum_{i=1}^L (\Delta' \mathcal{F}(C_q(e'))_{i,i+1}, \right. \\
&\quad \left. \sum_{j=1}^{L+1} (q^{d_\alpha h'_\alpha})_1 \otimes \dots \otimes (q^{d_\alpha h'_\alpha})_{j-1} \otimes (e'_\alpha)_j \otimes (q^{-d_\alpha h'_\alpha})_{j+1} \otimes \dots \otimes (q^{-d_\alpha h'_\alpha})_{L+1} \right] \\
&= \sum_{i=1}^L [(\Delta' \mathcal{F}(C_q(e'))_{i,i+1}, (e'_\alpha)_i \otimes (q^{-d_\alpha h'_\alpha})_{i+1} + (q^{d_\alpha h'_\alpha})_i \otimes (e'_\alpha)_{i+1})].
\end{aligned}$$

Using formulae (26) and (29) we get

$$[H_q, E'_\alpha] = \sum_{i=1}^L [(\Delta' \mathcal{F}(C_q(e'))_{i,i+1}, \Delta'(e'_\alpha))_{i,i+1}] = 0.$$

Similarly we have

$$\begin{aligned}
[H_q, F'_\alpha] &= \left[\sum_{i=1}^L (\Delta' \mathcal{F}(C_q(e'))_{i,i+1}, \right. \\
&\quad \left. \sum_{j=1}^{L+1} (q^{d_\alpha h'_\alpha})_1 \otimes \dots \otimes (q^{d_\alpha h'_\alpha})_{j-1} \otimes (f'_\alpha)_j \otimes (q^{-d_\alpha h'_\alpha})_{j+1} \otimes \dots \otimes (q^{-d_\alpha h'_\alpha})_{L+1} \right] \\
&= \sum_{i=1}^L \left[(\Delta' \mathcal{F}(C_q(e'))_{i,i+1}, (e'_\alpha)_i \otimes (q^{-d_\alpha h'_\alpha})_{i+1} + (q^{d_\alpha h'_\alpha})_i \otimes (f'_\alpha)_{i+1} \right] \\
&= \sum_{i=1}^L [(\Delta' \mathcal{F}(C_q(e')), \Delta'(f'_\alpha)]_{i,i+1} = 0.
\end{aligned}$$

Therefore H_q commutes with the generators of A_q for the chain. ■

The Hamiltonian (33) stands for a system with nearest neighbours interactions. Generally by using the coproduct operator Δ' we can construct models with $N+1$ ($N \leq L$) sites interactions:

$$H_q^N = \sum_{i=1}^{L-N+1} \mathcal{F}(\Delta'^N C_q(e'))_{i,i+1,\dots,i+N}, \quad (34)$$

Taking into account the relation

$$\Delta'^N ([\mathcal{F}(C_q(e')), q^{\pm d_\alpha h'_\alpha}]) = [\Delta'^N \mathcal{F}(C_q(e')), \underbrace{q^{\pm d_\alpha h'_\alpha} \otimes \dots \otimes q^{\pm d_\alpha h'_\alpha}}_{N \text{ times}}] = 0,$$

we can prove that the Hamiltonian H_q^N has the symmetry of the algebra A_q . In fact:

$$\begin{aligned}
[H_q^N, H'_\alpha] &= \sum_{i=1}^{L-N+1} \left[(\Delta'^N \mathcal{F}(C_q(e'))_{i,i+1,\dots,i+N}, \sum_{j=1}^{L+1} (h'_\alpha)_j \right] \\
&= \sum_{i=1}^{L-N+1} \left[(\Delta'^N \mathcal{F}(C_q(e'))_{i,i+1,\dots,i+N}, \sum_{j=i}^{i+N} (h'_\alpha)_j \right] \\
&= \sum_{i=1}^{L-N+1} [(\Delta'^N \mathcal{F}(C_q(e'))_{i,i+1,\dots,i+N}, \Delta'^N (h'_\alpha)]_{i,i+1,\dots,i+N} = 0;
\end{aligned}$$

$$\begin{aligned}
[H_q^N, E'_\alpha] &= \left[\sum_{i=1}^{L-N+1} (\Delta'^N \mathcal{F}(C_q(e'))_{i,i+1,\dots,i+N}, \right. \\
&\quad \left. \sum_{j=i}^{i+N} (q^{d_\alpha h'_\alpha})_i \otimes \dots \otimes (q^{d_\alpha h'_\alpha})_{j-1} \otimes (e'_\alpha)_j \otimes (q^{-d_\alpha h'_\alpha})_{j+1} \otimes \dots \otimes (q^{-d_\alpha h'_\alpha})_{i+N} \right] \\
&= \sum_{i=1}^{L-N+1} [(\Delta'^N \mathcal{F}(C_q(e'))_{i,i+1,\dots,i+N}, \Delta'^N (e'_\alpha)]_{i,i+1,\dots,i+N} = 0;
\end{aligned}$$

$$\begin{aligned}
[H_q^N, F'_\alpha] &= \left[\sum_{i=1}^{L-N+1} (\Delta'^N \mathbb{F}(C_q(e'))_{i,i+1,\dots,i+N}, \right. \\
&\quad \left. \sum_{j=i}^{i+N} (q^{d_\alpha h'_\alpha})_i \otimes \dots \otimes (q^{d_\alpha h'_\alpha})_{j-1} \otimes (f'_\alpha)_j \otimes (q^{-d_\alpha h'_\alpha})_{j+1} \otimes \dots \otimes (q^{-d_\alpha h'_\alpha})_{i+N} \right] \\
&= \sum_{i=1}^{L-N+1} [(\Delta'^N \mathbb{F}(C_q(e'))_{i,i+1,\dots,i+N}, \Delta'^N(f'_\alpha)]_{i,i+1,\dots,i+N} = 0.
\end{aligned}$$

The Hamiltonian system (33) is expressed by the quantum algebraic generators $e' = (h'_\alpha, e'_\alpha, f'_\alpha)$. Assume now that $e \rightarrow e'(e)$ is an algebraic map from A to A_q (we remark that for rank one algebras, both classical and quantum algebraic maps can be discussed in terms of the two dimensional manifolds related to the algebras, see [10]). We then have

$$H_q^N = \sum_{i=1}^{L-N+1} \mathbb{F}(\Delta'^N C_q(e'(e)))_{i,i+1,\dots,i+N}. \quad (35)$$

In this way we obtain Hamiltonian systems having quantum algebraic symmetry but expressed in terms of the usual Lie algebraic generators $\{e_\alpha\}$.

3 Integrable Models with A_n Symmetry

3.1 Quantum Yang-Baxter Equation

The quantum Yang-Baxter equation (QYBE) [11] is the master equation for integrable models in statistical mechanics. It plays an important role in a variety of problems in theoretical physics such as exactly soluble models (like the six and eight vertex models) in statistical mechanics [12], integrable model field theories [13], exact S-matrix theoretical models [14], two dimensional field theories involving fields with intermediate statistics [15], conformal field theory and quantum groups [9]. In this section we will investigate the integrability of the chain models having a certain algebraic symmetry constructed in section 2. We also present a series of solutions of the QYBE from the construction of integrable models.

Let V be a complex vector space and R be the solution of QYBE without spectral parameters, see e.g. [9]. Then R takes values in $\text{End}_{\mathfrak{d}}(V \otimes V)$. The QYBE is

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \quad (36)$$

Here R_{ij} denotes the matrix on the complex vector space $V \otimes V \otimes V$, acting as R on the i -th and the j -th components and as the identity on the other components.

Let $\check{R} = Rp$, p as in (6). Then the QYBE (36) becomes

$$\check{R}_{12}\check{R}_{23}\check{R}_{12} = \check{R}_{23}\check{R}_{12}\check{R}_{23}, \quad (37)$$

where $\check{R}_{12} = \check{R} \otimes \mathbf{1}$, $\check{R}_{23} = \mathbf{1} \otimes \check{R}$ and $\mathbf{1}$ is the identity operator on V .

A chain model with nearest neighbours interactions having a (quantum mechanical) Hamiltonian of the form

$$H = \sum_{i=1}^L (\mathcal{H})_{i,i+1} \quad (38)$$

is said to be integrable if the rank-two tensor operator \mathcal{H} satisfies the QYBE relation (37), i.e.,

$$(\mathcal{H})_{12}(\mathcal{H})_{23}(\mathcal{H})_{12} = (\mathcal{H})_{23}(\mathcal{H})_{12}(\mathcal{H})_{23}, \quad (39)$$

where

$$(\mathcal{H})_{12} = \mathcal{H} \otimes \mathbf{1}, \quad (\mathcal{H})_{23} = \mathbf{1} \otimes \mathcal{H}.$$

The Hamiltonian system (38) satisfying relation (39) can in principle be exactly solved by the algebraic Bethe Ansatz method, see e.g. [1].

3.2 Integrable A_n Symmetric Chain Models

The integrability of the models having a certain algebraic symmetry presented in section 2 depends on the detailed representation of the corresponding symmetry algebra. In the following we investigate the integrability of chain models with nearest neighbours interactions and Lie algebraic symmetry A_n .

Let $(a_{\alpha\beta})$ be the Cartan matrix of the A_n algebra. In the Chevalley basis the algebra A_n is spanned by the generators $\{h_\alpha, e_\alpha, f_\alpha\}$, $\alpha = 1, 2, \dots, n$, with the following algebraic relations:

$$\begin{aligned} [h_\alpha, h_\beta] &= 0, \\ [h_\alpha, e_\beta] &= a_{\alpha\beta} e_\beta, \\ [h_\alpha, f_\beta] &= -a_{\alpha\beta} f_\beta, \\ [e_\alpha, f_\beta] &= \delta_{\alpha\beta} h_\alpha, \end{aligned} \quad (40)$$

together with the generators with respect to non simple roots,

$$e_{\alpha\ldots\beta\gamma} = [e_\alpha, \ldots, [e_\beta, e_\gamma]\ldots], \quad f_{\alpha\ldots\beta\gamma} = [f_\alpha, \ldots, [f_\beta, f_\gamma]\ldots]. \quad (41)$$

Let $E_{\alpha\beta}$ be an $(n+1) \times (n+1)$ matrix such that $(E_{\alpha\beta})_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta}$, i.e., the only non zero element of the matrix $E_{\alpha\beta}$ is 1 at row α and column β . Hence

$$E_{\alpha\beta}E_{\gamma\delta} = \delta_{\beta\gamma}E_{\alpha\delta} \quad (42)$$

and

$$[E_{\alpha\beta}, E_{\gamma\delta}] = \delta_{\beta\gamma}E_{\alpha\delta} - \delta_{\delta\alpha}E_{\beta\gamma}.$$

For the fundamental representation we take the basis of the algebra A_n as

$$\begin{aligned} h_\alpha &= E_{\alpha\alpha} - E_{\alpha+1, \alpha+1}, \quad \alpha = 1, 2, \ldots, n \\ \left. \begin{aligned} e &= \{E_{\alpha\beta}\} \\ f &= \{E_{\beta\alpha}\} \end{aligned} \right\} \quad \beta > \alpha = 1, 2, \ldots, n \end{aligned} \quad (43)$$

Both $\{e_\alpha\}$ and $\{f_\alpha\}$ have a total of $n(n+1)/2$ generators.

With respect to the basis (43), the Casimir operator of the algebra A_n is given by

$$\begin{aligned} C_{A_n} &= (n+1) \sum_{\alpha=1}^{n(n+1)/2} (e_\alpha f_\alpha + f_\alpha e_\alpha) + \sum_{\alpha=1}^n \alpha(n+1-\alpha) h_\alpha^2 \\ &\quad + \sum_{\alpha=1}^n \sum_{\beta=1}^{n-\alpha} 2\alpha(n+1-\alpha-\beta) h_\alpha h_{\alpha+\beta} - a, \end{aligned} \quad (44)$$

where a is an arbitrary real constant.

The coproduct operator Δ is given by

$$\begin{aligned} \Delta(\mathbf{1}) &= \mathbf{1} \otimes \mathbf{1} \\ \Delta(h_\alpha) &= h_\alpha \otimes \mathbf{1} + \mathbf{1} \otimes h_\alpha, \quad \alpha = 1, 2, \ldots, n \\ \left. \begin{aligned} \Delta(e_\beta) &= e_\beta \otimes \mathbf{1} + \mathbf{1} \otimes e_\beta \\ \Delta(f_\beta) &= f_\beta \otimes \mathbf{1} + \mathbf{1} \otimes f_\beta \end{aligned} \right\} \quad \beta = 1, 2, \ldots, n(n+1)/2, \end{aligned} \quad (45)$$

where the identity operator $\mathbf{1}$ is the $(n+1) \times (n+1)$ identity matrix.

By (44) and (45) we have

$$\begin{aligned} \Delta C_{A_n} &= C_{A_n} \otimes \mathbf{1} + \mathbf{1} \otimes C_{A_n} - a \mathbf{1} \otimes \mathbf{1} \\ &\quad + (n+1) \sum_{\alpha=1}^{n(n+1)/2} (e_\alpha \otimes f_\alpha + f_\alpha \otimes e_\alpha) + \sum_{\alpha=1}^n \alpha(n+1-\alpha) h_\alpha \otimes h_\alpha \\ &\quad + \sum_{\alpha=1}^n \sum_{\beta=1}^{n-\alpha} \alpha(n+1-\alpha-\beta) (h_\alpha \otimes h_{\alpha+\beta} + h_{\alpha+\beta} \otimes h_\alpha). \end{aligned} \quad (46)$$

It is easy to check that under the representation (43) C_{A_n} is equal to $n(n+2)\mathbf{1}$. Therefore the sum of the first two terms on the right hand side of (46) is $2n(n+2)\mathbf{1} \times \mathbf{1}$. In the following we take a in (46) to be $2n(n+2)$ so that the terms that are proportional to $(n+1)^2 \times (n+1)^2$ identity matrix will disappear in (46).

From (43) and (46) we have

$$\begin{aligned} \Delta C_{A_n} = & (n+1) \sum_{\alpha \neq \beta=1}^{n+1} E_{\alpha\beta} \otimes E_{\beta\alpha} \\ & + \sum_{\alpha=1}^n \alpha(n+1-\alpha)(E_{\alpha\alpha} - E_{\alpha+1,\alpha+1}) \otimes (E_{\alpha\alpha} - E_{\alpha+1,\alpha+1}) \\ & + \sum_{\alpha=1}^n \sum_{\beta=1}^{n-\alpha} \alpha(n+1-\alpha-\beta)[(E_{\alpha\alpha} - E_{\alpha+1,\alpha+1}) \otimes (E_{\alpha+\beta,\alpha+\beta} - E_{\alpha+\beta+1,\alpha+\beta+1}) \\ & + (E_{\alpha+\beta,\alpha+\beta} - E_{\alpha+\beta+1,\alpha+\beta+1}) \otimes (E_{\alpha\alpha} - E_{\alpha+1,\alpha+1})]. \end{aligned} \quad (47)$$

ΔC_{A_n} in (47) is an $(n+1)^2 \times (n+1)^2$ matrix. Its matrix representation is

$$\begin{aligned} (\Delta C_{A_n})_{\alpha\beta} = & \delta_{\alpha\beta}[(n+1)\delta_{\alpha,l(n+1)+l+1} - 1] \\ & + (n+1)[\delta_{\alpha,j(n+2)+k+2}\delta_{\beta,(j+1)(n+2)+k(n+1)} \\ & + \delta_{\beta,j(n+2)+k+2}\delta_{\alpha,(j+1)(n+2)+k(n+1)}], \end{aligned} \quad (48)$$

where $\alpha, \beta = 1, 2, \dots, (n+1)^2$, $l = 0, 1, \dots, n$, $j = 0, 1, \dots, n-1$, $k = 0, 1, \dots, n-j-1$, $\delta_{\alpha,j(n+2)+k+2} = 0$ if $\alpha \neq j(n+2) + k + 2$ for all possible values of j and k . Here we give explicitly, as examples, the matrix representations of ΔC_{A_n} for $n = 1, 2, 3$:

$$\Delta C_{A_1} = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & 2 & \cdot \\ \cdot & 2 & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad (49)$$

[illegible]

$$\Delta C_{A_3} = \begin{pmatrix} 3 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & -1 & . & . & 4 & . & . & . & . & . & . & . & . & . & . & . \\ . & . & -1 & . & . & . & . & . & 4 & . & . & . & . & . & . & . \\ . & . & . & -1 & . & . & . & . & . & . & . & . & 4 & . & . & . \\ . & 4 & . & . & -1 & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & 3 & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & -1 & . & . & 4 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & -1 & . & . & . & . & . & 4 & . & . \\ . & . & 4 & . & . & . & . & . & -1 & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 4 & . & . & -1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & 3 & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & -1 & . & . & 4 & . \\ . & . & . & 4 & . & . & . & . & . & . & . & . & -1 & . & . & . \\ . & . & . & . & . & . & . & 4 & . & . & . & . & . & -1 & . & . \\ . & . & . & . & . & . & . & . & . & . & . & 4 & . & . & -1 & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 3 \end{pmatrix},$$

(51)

where for simplicity \cdot stands for 0.

[Lemma 1]. ΔC_{A_n} satisfies the following relation

$$(\Delta C_{A_n})^2 + 2\Delta C_{A_n} - n(n+2)\mathbf{1} \otimes \mathbf{1} = 0. \quad (52)$$

[Proof]. From (48) we have

$$\begin{aligned}
[(\Delta C_{A_n})^2]_{\alpha\gamma} &= \sum_{\beta=1}^{(n+1)^2} (\Delta C_{A_n})_{\alpha\beta} (\Delta C_{A_n})_{\beta\gamma} \\
&= \sum_{\beta=1}^{(n+1)^2} [\delta_{\alpha\beta}((n+1)\delta_{\alpha,l(n+1)+l+1} - 1) + (n+1)(\delta_{\alpha,j(n+2)+k+2}\delta_{\beta,(j+1)(n+2)+k(n+1)} \\
&\quad + \delta_{\beta,j(n+2)+k+2}\delta_{\alpha,(j+1)(n+2)+k(n+1)})] \\
&\quad [\delta_{\beta\gamma}((n+1)\delta_{\beta,l'(n+1)+l'+1} - 1) + (n+1)(\delta_{\beta,j'(n+2)+k'+2}\delta_{\gamma,(j'+1)(n+2)+k'(n+1)} \\
&\quad + \delta_{\gamma,j'(n+2)+k'+2}\delta_{\beta,(j'+1)(n+2)+k'(n+1)})] \\
&= \delta_{\alpha\gamma}[(n+1)^2\delta_{\alpha,l(n+1)+l+1}\delta_{\gamma,l'(n+1)+l'+1} \\
&\quad - (n+1)(\delta_{\alpha,l(n+1)+l+1} + \delta_{\gamma,l'(n+1)+l'+1}) + 1] \\
&\quad - 2(n+1)[\delta_{\alpha,j(n+2)+k+2}\delta_{\gamma,(j+1)(n+2)+k(n+1)} + \delta_{\gamma,j(n+2)+k+2}\delta_{\alpha,(j+1)(n+2)+k(n+1)}] \\
&\quad + (n+1)^2[\delta_{\alpha,j(n+2)+k+2}\delta_{\gamma,j(n+2)+k+2} + \delta_{\alpha,(j+1)(n+2)+k(n+1)}\delta_{\gamma,(j+1)(n+2)+k(n+1)}] \\
&= \delta_{\alpha\gamma}[(n-1)^2\delta_{\alpha,l(n+1)+l+1}] \\
&\quad - 2(n+1)[\delta_{\alpha,j(n+2)+k+2}\delta_{\gamma,(j+1)(n+2)+k(n+1)} + \delta_{\gamma,j(n+2)+k+2}\delta_{\alpha,(j+1)(n+2)+k(n+1)}] \\
&\quad + (n+1)^2\delta_{\alpha\gamma}[\delta_{\alpha,j(n+2)+k+2} + \delta_{\alpha,(j+1)(n+2)+k(n+1)}] \\
&= -2(\Delta C_{A_n})_{\alpha\gamma} + (n+1)^2\delta_{\alpha\gamma}[\delta_{\alpha,j(n+2)+k+2} + \delta_{\alpha,(j+1)(n+2)+k(n+1)}] \\
&\quad + (n+1)^2\delta_{\alpha\gamma}\delta_{\alpha,l(n+1)+l+1} - \delta_{\alpha\gamma} \\
&= -2(\Delta C_{A_n})_{\alpha\gamma} + (n+1)^2\delta_{\alpha\gamma} - \delta_{\alpha\gamma} \\
&= -2(\Delta C_{A_n})_{\alpha\gamma} + n(n+2)\delta_{\alpha\gamma},
\end{aligned}$$

where the identity

$$\delta_{\alpha,l(n+1)+l+1} + \delta_{\alpha,j(n+2)+k+2} + \delta_{\alpha,(j+1)(n+2)+k(n+1)} = 1, \quad (53)$$

$l = 0, 1, \dots, n, j = 0, 1, \dots, n-1, k = 0, 1, \dots, n-j-1$, has been used. ■

[Lemma 2]. The coproduct of the A_n Casimir operator ΔC_{A_n} has the following properties:

$$\begin{aligned}
&(\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1}) \\
&- n[(\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1}) + (\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n})] \\
&+ (n^2 - 1)(\Delta C_{A_n} \otimes \mathbf{1}) + n^2(\mathbf{1} \otimes \Delta C_{A_n}) + n(1 - n^2)\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} = 0
\end{aligned} \quad (54)$$

and

$$\begin{aligned}
&(\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n}) \\
&- n[(\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1}) + (\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n})] \\
&+ (n^2 - 1)(\mathbf{1} \otimes \Delta C_{A_n}) + n^2(\Delta C_{A_n} \otimes \mathbf{1}) + n(1 - n^2)\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} = 0.
\end{aligned} \quad (55)$$

[Proof]. By using the representation of ΔC_{A_n} in (48) we have

$$\begin{aligned}
(\Delta C_{A_n} \otimes \mathbf{1})_{\alpha\beta} &= (\Delta C_{A_n})_{(\alpha-\gamma)/(n+1)+1, (\beta-\gamma)/(n+1)+1} \\
&= \delta_{\alpha\beta} [(n+1)\delta_{\alpha-\gamma, l(n+1)(n+2)} - 1] \\
&\quad + (n+1) [\delta_{\alpha-\gamma, (n+1)(j(n+2)+k+1)} \delta_{\beta-\gamma, (n+1)(j(n+2)+(k+1)(n+1))} \\
&\quad + \delta_{\beta-\gamma, (n+1)(j(n+2)+k+1)} \delta_{\alpha-\gamma, (n+1)(j(n+2)+(k+1)(n+1))}]
\end{aligned} \tag{56}$$

and

$$\begin{aligned}
(\mathbf{1} \otimes \Delta C_{A_n})_{\alpha\beta} &= (\Delta C_{A_n})_{\alpha-(n+1)^2(\gamma'-1), \beta-(n+1)^2(\gamma'-1)} \\
&= \delta_{\alpha\beta} [(n+1)\delta_{\alpha-(n+1)^2(\gamma'-1), l(n+1)+l+1} - 1] \\
&\quad + (n+1) [\delta_{\alpha-(n+1)^2(\gamma'-1), j(n+2)+k+2} \delta_{\beta-(n+1)^2(\gamma'-1), (j+1)(n+2)+k(n+1)} \\
&\quad + \delta_{\beta-(n+1)^2(\gamma'-1), j(n+2)+k+2} \delta_{\alpha-(n+1)^2(\gamma'-1), (j+1)(n+2)+k(n+1)}],
\end{aligned} \tag{57}$$

where $\alpha, \beta = 1, \dots, (n+1)^3$, $l = 0, 1, \dots, n$, $j = 0, 1, \dots, n-1$ and $k = 0, 1, \dots, n-j-1$ as in formula (48), $\gamma = 1, \dots, n+1$ such that $(\alpha-\gamma)/(n+1)$ and $(\beta-\gamma)/(n+1)$ in (56) are integers and $\gamma' = 1, \dots, n+1$ in (57).

Using the formulae (56) and (57) one can get (54) and (55) from straightforward calculations. ■

From Theorem 1 we know that the following Hamiltonian is invariant under A_n

$$H = \sum_{i=1}^L \mathcal{F}(\Delta C_{A_n})_{i, i+1}. \tag{58}$$

For the given representation (43) of A_n the integrability of (48) depends on the form of the entire function \mathcal{F} . Due to the relation (52) in Lemma 1, $(\Delta C_{A_n})^l$, $l \geq 2$, can be expressed as $c\Delta C_{A_n} + c'\mathbf{1} \otimes \mathbf{1}$ for some real constants c and c' . Therefore $\mathcal{F}(\Delta C_{A_n})$ is a polynomial in ΔC_{A_n} up to powers of order two.

[Theorem 3]. The following A_n invariant Hamiltonian is integrable

$$\begin{aligned}
H_{A_n} &= \sum_{i=1}^L (\mathcal{H})_{i, i+1} = \sum_{i=1}^L (\Delta C_{A_n} + 1)_{i, i+1} \\
&= \sum_{i=1}^L \left[(n+1) \sum_{\alpha=1}^{n(n+1)/2} ((e_\alpha)_i (f_\alpha)_{i+1} + (f_\alpha)_i (e_\alpha)_{i+1}) + \sum_{\alpha=1}^n \alpha(n+1-\alpha) (h_\alpha)_i (h_\alpha)_{i+1} \right. \\
&\quad \left. + \sum_{\alpha=1}^n \sum_{\beta=1}^{n-\alpha} \alpha(n+1-\alpha-\beta) ((h_\alpha)_i (h_{\alpha+\beta})_{i+1} + (h_{\alpha+\beta})_i (h_\alpha)_{i+1}) \right] + L,
\end{aligned} \tag{59}$$

where $\mathcal{H} = \Delta C_{A_n} + 1$ and the number 1 should be understood as the identity operator, $\mathbf{1} \otimes \mathbf{1}$, on the tensor space $H_1 \otimes \dots \otimes H_{L+1}$ ($L+1$ is the number of lattice sites of the chain).

[Proof]. What we have to prove is that \mathcal{H} satisfies the QYBE (39), i.e.,

$$(\mathcal{H})_{12}(\mathcal{H})_{23}(\mathcal{H})_{12} = (\mathcal{H})_{23}(\mathcal{H})_{12}(\mathcal{H})_{23},$$

where

$$(\mathcal{H})_{12} = (\Delta C_{A_n} + 1) \otimes \mathbf{1}, \quad (\mathcal{H})_{23} = \mathbf{1} \otimes (\Delta C_{A_n} + 1). \quad (60)$$

From (60) we have

$$\begin{aligned} & (\mathcal{H})_{12}(\mathcal{H})_{23}(\mathcal{H})_{12} \\ &= (\mathbf{1} \otimes \Delta C_{A_n} + \Delta C_{A_n} \otimes \mathbf{1} + (\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n}) + \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1})(\mathcal{H})_{12} \\ &= (\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1}) + (\Delta C_{A_n} \otimes \mathbf{1})(\Delta C_{A_n} \otimes \mathbf{1}) \\ &\quad + (\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1}) + (\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n}) \\ &\quad + 2\Delta C_{A_n} \otimes \mathbf{1} + \mathbf{1} \otimes \Delta C_{A_n} + \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} \end{aligned}$$

and

$$\begin{aligned} & (\mathcal{H})_{23}(\mathcal{H})_{12}(\mathcal{H})_{23} \\ &= (\mathcal{H})_{23}(\mathbf{1} \otimes \Delta C_{A_n} + \Delta C_{A_n} \otimes \mathbf{1} + (\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n}) + \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}) \\ &= (\mathbf{1} \otimes \Delta C_{A_n})(\mathbf{1} \otimes \Delta C_{A_n}) + (\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1}) \\ &\quad + (\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n}) + (\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n}) \\ &\quad + 2(\mathbf{1} \otimes \Delta C_{A_n}) + (\Delta C_{A_n} \otimes \mathbf{1}) + \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}. \end{aligned}$$

Hence

$$(\mathcal{H})_{12}(\mathcal{H})_{23}(\mathcal{H})_{12} - (\mathcal{H})_{23}(\mathcal{H})_{12}(\mathcal{H})_{23} = I + II + III, \quad (61)$$

where

$$\begin{aligned} I &= (\Delta C_{A_n} \otimes \mathbf{1})(\Delta C_{A_n} \otimes \mathbf{1}) - (\mathbf{1} \otimes \Delta C_{A_n})(\mathbf{1} \otimes \Delta C_{A_n}), \\ II &= (\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1}) - (\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n}), \\ III &= \Delta C_{A_n} \otimes \mathbf{1} - \mathbf{1} \otimes \Delta C_{A_n}. \end{aligned}$$

Using (52) we have

$$\begin{aligned}
I &= (\Delta C_{A_n} \otimes \mathbf{1})(\Delta C_{A_n} \otimes \mathbf{1}) - (\mathbf{1} \otimes \Delta C_{A_n})(\mathbf{1} \otimes \Delta C_{A_n}) \\
&= (\Delta C_{A_n})^2 \otimes \mathbf{1} - \mathbf{1} \otimes (\Delta C_{A_n})^2 \\
&= -(2\Delta C_{A_n} - n(n+2)\mathbf{1} \otimes \mathbf{1}) \otimes \mathbf{1} + \mathbf{1} \otimes (2\Delta C_{A_n} - n(n+2)\mathbf{1} \otimes \mathbf{1}) \\
&= -2(\Delta C_{A_n} \otimes \mathbf{1} - \mathbf{1} \otimes \Delta C_{A_n}).
\end{aligned}$$

Therefore

$$I + II = \mathbf{1} \otimes \Delta C_{A_n} - \Delta C_{A_n} \otimes \mathbf{1}. \quad (62)$$

By Lemma 2 we get

$$\begin{aligned}
III &= (\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1}) - (\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n}) \\
&= \Delta C_{A_n} \otimes \mathbf{1} - \mathbf{1} \otimes \Delta C_{A_n}.
\end{aligned}$$

Therefore

$$(\mathcal{H})_{12}(\mathcal{H})_{23}(\mathcal{H})_{12} - (\mathcal{H})_{23}(\mathcal{H})_{12}(\mathcal{H})_{23} = I + II + III = 0.$$

■

3.3 Temperley-Lieb Algebraic Structures and Representations

An L -state TL algebra is described by the elements e_i , $i = 1, 2, \dots, L$, satisfying the TL algebraic relations [16],

$$\begin{aligned}
e_i e_{i \pm 1} e_i &= e_i, \\
e_i e_j &= e_j e_i, \quad \text{if } |i - j| \geq 2,
\end{aligned} \quad (63)$$

and

$$e_i^2 = \beta e_i, \quad (64)$$

where β is a complex constant and $i = 1, 2, \dots, L$.

In this section we indicate that there is a TL algebraic structure related to the integrable chain model (59), in the sense that the model gives a representation of the TL algebra. We suppose that the representation of an L -state TL algebra on an $L + 1$ chain is of the following form,

$$e_i = \mathbf{1}_1 \otimes \mathbf{1}_2 \otimes \dots \otimes \mathbf{1}_{i-1} \otimes E \otimes \mathbf{1}_{i+2} \otimes \dots \otimes \mathbf{1}_{L+1}, \quad (65)$$

where $\mathbf{1}$ is the $(n+1) \times (n+1)$ identity matrix as in section 3.2 and E is a $(n+1)^2 \times (n+1)^2$ matrix. According to formulae (64) and (63) E should satisfy

$$E^2 = \beta E. \quad (66)$$

$$\begin{aligned} (E \otimes \mathbf{1})(\mathbf{1} \otimes E)(E \otimes \mathbf{1}) &= E \otimes \mathbf{1}, \\ (\mathbf{1} \otimes E)(E \otimes \mathbf{1})(\mathbf{1} \otimes E) &= \mathbf{1} \otimes E. \end{aligned} \quad (67)$$

[Theorem 4]. For a given representation of the TL algebra of the form (65) with E satisfying (66) and (67) we have that

$$\check{R} = E + \frac{-\beta \pm \sqrt{\beta^2 - 4}}{2} \mathbf{1} \otimes \mathbf{1} \quad (68)$$

is a solution of the QYBE (37).

[Proof]. For simplicity we set $c = (-\beta \pm \sqrt{\beta^2 - 4})/2$. Substituting (68) into equation (37) and using relations (66) and (67) we get

$$\begin{aligned} &\check{R}_{12}\check{R}_{23}\check{R}_{12} - \check{R}_{23}\check{R}_{12}\check{R}_{23} \\ &= (E \otimes \mathbf{1})(\mathbf{1} \otimes E)(E \otimes \mathbf{1}) - (\mathbf{1} \otimes E)(E \otimes \mathbf{1})(\mathbf{1} \otimes E) \\ &\quad + c(E^2 \otimes \mathbf{1} - \mathbf{1} \otimes E^2) + c^2(E \otimes \mathbf{1} - \mathbf{1} \otimes E) \\ &= (E \otimes \mathbf{1})(\mathbf{1} \otimes E)(E \otimes \mathbf{1}) - (\mathbf{1} \otimes E)(E \otimes \mathbf{1})(\mathbf{1} \otimes E) + (c\beta + c^2)(E \otimes \mathbf{1} - \mathbf{1} \otimes E) \\ &= (c\beta + c^2 + 1)(E \otimes \mathbf{1} - \mathbf{1} \otimes E) = 0. \end{aligned}$$

■

In general however the converse does not hold, i.e., for a given solution \check{R} of the QYBE (37), there does not necessarily exist a TL algebraic representation of the form (65) with $E = a\check{R} + b$ satisfying (66) and (67) for any constants a and b . Nevertheless the solutions \mathcal{H} of the QYBE in our A_n symmetric integrable model (59) do give rise to TL algebraic representations in the following sense:

[Theorem 5]. The following $(n+1)^2 \times (n+1)^2$ matrix

$$E = -\frac{\mathcal{H}}{n+1} + \mathbf{1} \otimes \mathbf{1} \quad (69)$$

gives the L -state TL algebraic representation (65) with $\beta = 2$.

[Proof]. What we should check is that E in (69) satisfies equations (66) and (67). By

Lemma 1 we have

$$\begin{aligned}
E^2 &= \left(-\frac{\mathcal{H}}{n+1} + \mathbf{1} \otimes \mathbf{1}\right)^2 = \left(-\frac{\Delta C_{A_n} + \mathbf{1} \otimes \mathbf{1}}{n+1} + \mathbf{1} \otimes \mathbf{1}\right)^2 \\
&= \frac{(\Delta C_{A_n})^2 - 2n\Delta C_{A_n} + n^2\mathbf{1} \otimes \mathbf{1}}{(n+1)^2} \\
&= \frac{(-2(n+1)\Delta C_{A_n} + 2n(n+1)\mathbf{1} \otimes \mathbf{1})}{(n+1)^2} \\
&= \beta E = 2E,
\end{aligned}$$

i.e., $\beta = 2$.

From Lemma 1 and (54) in Lemma 2 we get

$$\begin{aligned}
&(E \otimes \mathbf{1})(\mathbf{1} \otimes E)(E \otimes \mathbf{1}) \\
&= \left(\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} - \frac{\mathcal{H} \otimes \mathbf{1}}{n+1}\right) \left(\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} - \frac{\mathbf{1} \otimes \mathcal{H}}{n+1}\right) \left(\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} - \frac{\mathcal{H} \otimes \mathbf{1}}{n+1}\right) \\
&= \frac{-1}{(n+1)^3} [(\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1}) \\
&\quad - n((\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n}) + (\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1})) \\
&\quad - n(\Delta C_{A_n})^2 \otimes \mathbf{1} + 2n^2\Delta C_{A_n} \otimes \mathbf{1} + n^2\mathbf{1} \otimes \Delta C_{A_n} - n^3\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}] \\
&= \frac{-1}{(n+1)^3} [(\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1}) \\
&\quad - n((\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n}) + (\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1})) \\
&\quad + 2n(n+1)\Delta C_{A_n} \otimes \mathbf{1} + n^2\mathbf{1} \otimes \Delta C_{A_n} - 2(n^3 + n^2)\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}] \\
&= \frac{-1}{(n+1)^3} [(n+1)^2\Delta C_{A_n} \otimes \mathbf{1} - n(n+1)^2\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}] \\
&= \frac{n\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} - \Delta C_{A_n} \otimes \mathbf{1}}{(n+1)} = E \otimes \mathbf{1}.
\end{aligned}$$

By using Lemma 1 and formula (55) in Lemma 2 we then conclude that

$$\begin{aligned}
& (\mathbf{1} \otimes E)(E \otimes \mathbf{1})(\mathbf{1} \otimes E) \\
&= (\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} - \frac{\mathbf{1} \otimes \mathcal{H}}{n+1})(\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} - \frac{\mathcal{H} \otimes \mathbf{1}}{n+1})(\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} - \frac{\mathbf{1} \otimes \mathcal{H}}{n+1}) \\
&= \frac{-1}{(n+1)^3} [(\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n}) \\
&\quad - n((\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n}) + (\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1})) \\
&\quad - n\mathbf{1} \otimes (\Delta C_{A_n})^2 + 2n^2\mathbf{1} \otimes \Delta C_{A_n} + n^2\Delta C_{A_n} \otimes \mathbf{1} - n^3\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}] \\
&= \frac{-1}{(n+1)^3} [(\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1}) \\
&\quad - n((\Delta C_{A_n} \otimes \mathbf{1})(\mathbf{1} \otimes \Delta C_{A_n}) + (\mathbf{1} \otimes \Delta C_{A_n})(\Delta C_{A_n} \otimes \mathbf{1})) \\
&\quad + 2n(n+1)\mathbf{1} \otimes \Delta C_{A_n} + n^2\Delta C_{A_n} \otimes \mathbf{1} - 2(n^3 + n^2)\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}] \\
&= \frac{-1}{(n+1)^3} [(n+1)^2\mathbf{1} \otimes \Delta C_{A_n} - n(n+1)^2\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}] \\
&= \frac{n\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1} - \mathbf{1} \otimes \Delta C_{A_n}}{(n+1)} = \mathbf{1} \otimes E.
\end{aligned}$$

■

From (69) we see that the Hamiltonian of the A_n symmetric integrable chain model (59) can be expressed by the TL algebraic elements

$$H_{A_n} = \sum_{i=1}^L (\mathcal{H})_{i,i+1} = \sum_{i=1}^L (n+1)(-E+1)_{i,i+1} = \sum_{i=1}^L (n+1)e_i + (n+1)L, \quad (70)$$

with e_i as in (65) and E as in (69). Hence instead of the algebraic Bethe Ansatz method, the energy spectrum of H_{A_n} can also be studied by using the properties of the TL algebra [17] (for the case of Heisenberg spin chain model, $n=1$, see [18]).

4 Integrable Models and Stationary Markov Chains

4.1 Stationary Markov Chains

We first briefly recall some concepts of the theory of Markov chains (for a detailed mathematical description of Markov chains, we refer to [19]). Let Ω denote the sample space (the set of all possible outcomes of an experiment). If Ω is a finite or countably infinite sample space and if P is a probability measure defined on the σ -algebra of all subsets of

Ω , then the pair (Ω, P) is called a probability space. A subset A of Ω is then called an event with probability $P(A)$.

A function $X \equiv X(\omega)$, $\omega \in \Omega$, that maps a sample space into the real numbers is called a random variable. A stochastic process is a family $(X_t)_{t \in I}$, I a certain index set, of random variables defined on some sample space Ω . If I is countable, i.e., $I \in \mathbb{N}$, the process is denoted by X_1, X_2, \dots and called a discrete-time process. If $I = \mathbb{R}_+$, then the process is denoted by $\{X_t\}_{t \geq 0}$ and called a continuous-time process.

The ranges of X (a subset of real numbers) is called the state space. In what follows we consider the case where the state space S is countable or finite. In this case the related stochastic process is called a (stochastic or random) chain.

Let (Ω, P) be a probability space in above sense and E, F be two subsets of Ω . We denote by $P(E|F)$ the (conditional) probability of E given that F has occurred. A discrete-time stochastic process $\{X_i\}$, $i = 1, 2, \dots$ with state space $S = \mathbb{N}$ is said to satisfy the Markov property if for every l and all states i_1, i_2, \dots, i_l it is true that

$$P[X_l = i_l | X_{l-1} = i_{l-1}, X_{l-2} = i_{l-2}, \dots, X_1 = i_1] = P[X_l = i_l | X_{l-1} = i_{l-1}],$$

i.e., the values of X_{l-2}, \dots, X_1 in no way affect the value of X_l , given the value of X_{l-1} . Such a discrete-time process is called a Markov chain. It is said to be stationary if the probability of going from one state to another is independent of the time at which the transition is being made. That is, for all states i and j ,

$$P[X_l = j | X_{l-1} = i] = P[X_{l+k} = j | X_{l+k-1} = i]$$

for $k = -(l-1), -(l-2), \dots, -1, 0, 1, 2, \dots$. In this case we set $p_{ij} \equiv P[X_l = j | X_{l-1} = i]$ and call p_{ij} the transition probability for going from state i to j .

For a discrete time stationary Markov chain $\{X_i\}$, $i \in \mathbb{N}$, with a finite state space $S = \{1, 2, 3, \dots, m\}$, there are m^2 transition probabilities $\{p_{ij}\}$, $i, j = 1, 2, \dots, m$. $P = (p_{ij})$ is called the transition matrix corresponding to the discrete-time stationary Markov chain $\{X_i\}$. The transition matrix P has the following properties:

$$p_{ij} \geq 0, \quad \sum_{i=1}^m p_{ij} = 1, \quad i, j = 1, 2, \dots, m. \quad (71)$$

Any square matrix that satisfies condition (71) is called a stochastic matrix.

A continuous-time stochastic process, $\{X_t\}_{t \in \mathbb{R}_+}$ is said to satisfy the Markov property if for all times $t_0 < t_1 < \dots < t_l < t$ and for all l it is true that

$$P[X_t = j | X_{t_0} = i_0, X_{t_1} = i_1, \dots, X_{t_l} = i_l] = P[X_t = j | X_{t_l} = i_l].$$

Such a process is called a continuous-time Markov chain. It is said to be stationary if for every i and j the transition function, $P[X_{t+h} = j | X_t = i]$, is independent of t . In this case $P(t) = P(X_{t=j} | X_0 = i)$ is a semigroup (e.g. on $l^2(S)$), called transition semigroup associated with the Markov chain. Its generator $Q = (q_{ij})$ has the properties:

$$q_{ij} \geq 0, \quad i \neq j, \quad q_{ii} = -\sum_{i \neq j} q_{ij}, \quad i, j = 1, 2, \dots, m \quad (72)$$

and is called an intensity matrix. Vice versa, any Q (satisfying (72) and properly defined as a closed operator when S is infinite) gives rise to a unique continuous-time transition semigroup, $P(t) = e^{Qt}$, $t \geq 0$, which can be interpreted as transition semigroup associated to a certain Markov chain (with state space S) [19].

The properties of Markov chains are determined by the transition matrix P for discrete-time stochastic process and the intensity matrix Q for continuous-time stochastic processes. If the eigenvalues and eigenstates of P and Q are known, then exact results related to the stochastic processes, such as time-dependent averages and correlations, can be obtained.

Now we consider a chain (in the algebraic sense of sections 1-3) with $L + 1$ sites. To every site i of the chain we associate $n + 1$ states described by the variable τ_i taking $n + 1$ integer values,

$$\tau_i \equiv (\tau_i^0 = 0, \tau_i^1 = 1, \tau_i^2 = 2, \dots, \tau_i^n = n), \quad (73)$$

(conventionally a vacancy at site i is associated with the state 0). We associate to the lattice site i of the algebraic chain a Hilbert space of dimension $n + 1$. The state space of the algebraic chain is then finite and has a total of $(n + 1)^{L+1}$ states.

By definition, for an integrable chain model with Hamiltonian H one has exact solutions for the eigenvalues and eigenstates of H . The system remains integrable if one adds to H a constant term c and multiplies H by a constant factor c' . Moreover the eigenvalues of H will not be changed if one changes the local basis, i.e., the following Hamiltonian H' ,

$$H' = BHB^{-1}, \quad B = \otimes_{i=1}^{L+1} B_i, \quad (74)$$

where $B_i \equiv b$ and b is an $(n+1) \times (n+1)$ non singular matrix, has the same eigenvalues as H . Therefore if an integrable chain model with Hamiltonian H can be transformed by B (modulo c, c') into a stochastic matrix P , in the sense that

$$P = B(c'H + c\mathbb{1})B^{-1}, \quad (75)$$

where $\mathbb{1}$ is the $(n+1)^{L+1} \times (n+1)^{L+1}$ identity matrix, B as in (74), such that P satisfies (71) (with $m = (n+1)^{L+1}$), then P defines a discrete-time Markov chain and the related stochastic process can be simply studied by using the properties of the related integrable model with Hamiltonian H .

And if an integrable chain model with Hamiltonian H can be transformed (modulo c, c') into an intensity matrix Q , in the sense that

$$Q = B(c'H + c\mathbb{1})B^{-1} \quad (76)$$

with Q satisfying (72) (with $m = (n+1)^{L+1}$), then Q determines a continuous-time Markov chain and its properties can also be obtained by using the results of the related integrable Hamiltonian H .

In the following we discuss the question of whether the integrable models obtained in the way presented in this paper could be transformed into stationary Markov chains through transformations of the forms (75) or (76). Of course a stochastic matrix P (i.e. as given by (71)) can not in general be transformed into an intensity matrix Q (as given by (72)) by the spectrum preserved similarity transformation, $P = B(c'Q + c\mathbb{1})B^{-1}$. That is, an integrable chain model with Hamiltonian H that gives rise to a discrete-time Markov chain by the transformation (75) will in general not give rise to a continuous time Markov chain by (76), and vice versa.

4.2 Discrete-time Markov Chains Related to A_n Symmetric Integrable Models

We first note that for an integrable chain model with Hamiltonian $H = \sum_{i=1}^L h_{i,i+1}$ and $(n+1)$ states at every site $i, i = 1, 2, \dots, L+1$, if the sum of the elements in any row of the $(n+1)^2 \times (n+1)^2$ matrix h is $1/L$, then the sum of the elements in any row of the matrix H is 1. Hence if under the following transformation $h \rightarrow h'$ given by

$$h' = (b \otimes b)(c'h + c\mathbf{1} \otimes \mathbf{1})(b^{-1} \otimes b^{-1}), \quad (77)$$

the sum of the elements in any row of h' is $1/L$ and $(h')_{\alpha,\beta} \geq 0$, $\alpha, \beta = 1, 2, \dots, (n+1)^2$, for some real constants c', c and a non singular $(n+1) \times (n+1)$ matrix b , then $P = \sum_{i=1}^L h'_{i,i+1}$ defines a stationary discrete-time Markov chain. P has the same eigenvalue spectrum (shifted by a constant) as the spectrum of the integrable model with Hamiltonian H . If P is invariant under a certain algebra A , we call the Markov chain A symmetric.

[Theorem 6]. The following matrix

$$\begin{aligned}
P_{A_n} &= \frac{1}{L(n+1)} H_{A_n} = \frac{1}{L(n+1)} \sum_{i=1}^L (\Delta C_{A_n} + \mathbf{1} \otimes \mathbf{1})_{i,i+1} \\
&= \frac{1}{L(n+1)} \sum_{i=1}^L \left[(n+1) \sum_{\alpha=1}^{n(n+1)/2} ((e_\alpha)_i (f_\alpha)_{i+1} + (f_\alpha)_i (e_\alpha)_{i+1}) \right. \\
&\quad + \sum_{\alpha=1}^n \alpha(n+1-\alpha) (h_\alpha)_i (h_\alpha)_{i+1} \\
&\quad \left. + \sum_{\alpha=1}^n \sum_{\beta=1}^{n-\alpha} \alpha(n+1-\alpha-\beta) ((h_\alpha)_i (h_{\alpha+\beta})_{i+1} + (h_{\alpha+\beta})_i (h_\alpha)_{i+1}) \right] + \frac{1}{(n+1)}
\end{aligned} \tag{78}$$

defines a stationary discrete-time A_n symmetric Markov chain.

[Proof]. I. Set $h' \equiv \frac{1}{L(n+1)} (\Delta C_{A_n} + \mathbf{1} \otimes \mathbf{1})$. Then

$$P_{A_n} = \sum_{i=1}^L h'_{i,i+1}. \tag{79}$$

From formula (48) we have

$$\begin{aligned}
(h')_{\alpha\beta} &= \frac{1}{L(n+1)} (\Delta C_{A_n} + \mathbf{1} \otimes \mathbf{1})_{\alpha\beta} \\
&= \frac{1}{L(n+1)} [\delta_{\alpha\beta} [(n+1)\delta_{\alpha, l(n+1)+l+1} - 1] \\
&\quad + (n+1) [\delta_{\alpha, j(n+2)+k+2} \delta_{\beta, (j+1)(n+2)+k(n+1)} \\
&\quad + \delta_{\beta, j(n+2)+k+2} \delta_{\alpha, (j+1)(n+2)+k(n+1)}] + \delta_{\alpha\beta}] \\
&= \frac{1}{L} [\delta_{\alpha\beta} \delta_{\alpha, l(n+1)+l+1} + \delta_{\alpha, j(n+2)+k+2} \delta_{\beta, (j+1)(n+2)+k(n+1)} \\
&\quad + \delta_{\beta, j(n+2)+k+2} \delta_{\alpha, (j+1)(n+2)+k(n+1)}] \geq 0.
\end{aligned} \tag{80}$$

Therefore $(P_{A_n})_{\alpha\beta} \geq 0$, $\alpha, \beta = 1, 2, \dots, (n+1)^2$.

II. By using the identity (53), we get

$$\begin{aligned}
\sum_{\beta=1}^{(n+1)^2} (h')_{\alpha\beta} &= \frac{1}{L(n+1)} \sum_{\beta=1}^{(n+1)^2} [(n+1)\delta_{\alpha\beta}\delta_{\alpha, l(n+1)+l+1} \\
&\quad + (n+1)[\delta_{\alpha, j(n+2)+k+2}\delta_{\beta, (j+1)(n+2)+k(n+1)} \\
&\quad + \delta_{\beta, j(n+2)+k+2}\delta_{\alpha, (j+1)(n+2)+k(n+1)}]] \\
&= \frac{1}{L(n+1)}(n+1) = \frac{1}{L}.
\end{aligned}$$

Hence the sum of the elements of any row of the matrix P_{A_n} is one, i.e., $\sum_{\beta=1}^{(n+1)^{L+1}} (P_{A_n})_{\alpha\beta} = \sum_{\beta=1}^{(n+1)^{L+1}} (\sum_{i=1}^L h'_{i, i+1})_{\alpha\beta} = 1$.

III. As H_{A_n} is invariant under A_n , $P_{A_n} = \frac{H_{A_n}}{L(n+1)}$ is obviously invariant under A_n and has the same spectrum as H_{A_n} .

By the definition (71) P_{A_n} is the transition matrix of a stationary discrete-time A_n symmetric Markov chain. ■

We give some discussions on the properties of the stationary discrete-time A_n symmetric Markov chain associated with the stochastic matrix P_{A_n} . The state space of this Markov chain is, $S = (1, 2, \dots, (n+1)^{L+1})$, which corresponds to $(n+1)^{L+1}$ states,

$$(\tau_0 \otimes \tau_1 \otimes \dots \otimes \tau_n), \quad (81)$$

τ_i as in (73), of the algebraic chain with $L+1$ lattice sites.

The properties of a Markov chain are determined by the transition matrix $P = (p_{ij})$. A subset C of the state space S is called closed if $p_{ij} = 0$ for all $i \in C$ and $j \notin C$. If a closed set consists of a single state, then that state is called an absorbing state. A Markov chain is called irreducible if there exists no nonempty closed set other than S itself.

From formula (80) we have

$$\begin{aligned}
(h')_{\alpha\alpha} &= (h')_{(n+1)^2, (n+1)^2} = \frac{1}{L}, \quad (h')_{\alpha\beta} = (h')_{\beta\alpha} = 0, \quad \beta \neq \alpha, \\
\alpha &= l(n+1) + l + 1, \quad l = 0, 1, \dots, n.
\end{aligned} \quad (82)$$

Let

$$S_0 = \left(\alpha | \alpha = l \frac{(n+1)((n+1)^L - 1) + n}{n} + 1 \right), \quad l = 0, 1, \dots, n, \quad (83)$$

be a subset of the state space S . From formula (79), with

$$(h')_{i, i+1} = \mathbf{1}_1 \otimes \mathbf{1}_2 \otimes \dots \otimes \mathbf{1}_{i-1} \otimes h' \otimes \mathbf{1}_{i+2} \otimes \dots \otimes \mathbf{1}_{L+1},$$

we get

$$(P_{A_n})_{\alpha\alpha} = 1, \quad (P_{A_n})_{\beta\alpha} = (P_{A_n})_{\alpha\beta} = 0, \quad \beta \neq \alpha, \quad \alpha \in S_0. \quad (84)$$

Therefore the $n + 1$ states in S_0 are absorbing states of the Markov chain P_{A_n} . This chain is by definition reducible. For a reducible Markov chain the “long time” probability distribution, if it exists, may depend on the initial conditions, i.e., $\lim_{l \rightarrow \infty} (P_{A_n})_{\gamma\beta}^l$ may depend on γ . From the properties (84) of P_{A_n} , we see that if the Markov chain P_{A_n} is initially at one of the states $\alpha \in S_0$, it will remain in that state α forever. These $n + 1$ absorbing states correspond to the states of the algebraic chain through (81). For instance, the states 1 and $(n + 1)^{L+1}$ in S correspond to the states $(0, 0, \dots, 0)$ (all the sites of the algebraic chain are at state 0) and (n, n, \dots, n) (all the sites of the algebraic chain are at state n).

4.3 Continuous-time Markov Chains Related to A_n Symmetric Integrable Models

For an integrable chain model with Hamiltonian $H = \sum_{i=1}^L h_{i,i+1}$ and with $(n + 1)$ states at every site of the chain, if the sum of the elements in any column of the matrix h is 0, then the sum of the elements in any column of the matrix H is also 0. Hence if under the following transformation $h \rightarrow h''$ with:

$$h'' = (b \otimes b)(c'h + c\mathbf{1} \otimes \mathbf{1})(b^{-1} \otimes b^{-1}), \quad (85)$$

the sum of the elements in any column of h'' is 0 and $(h'')_{\alpha,\beta} \geq 0$, $\alpha \neq \beta = 1, 2, \dots, (n + 1)^2$, for some real constants c', c and a non singular $(n + 1) \times (n + 1)$ matrix b , then $Q = \sum_{i=1}^L h''_{i,i+1}$ is the intensity matrix for some stationary continuous-time Markov chain. Q has the same eigenvalue spectrum (shifted by a constant) as the spectrum of the Hamiltonian H . We call the Markov chain A symmetric if Q is invariant under the algebra A .

[Theorem 7]. The following matrix Q is the intensity matrix of a stationary continuous-time Markov chain,

$$\begin{aligned}
Q_{A_n} &= H_{A_n} - (n+1)L = \sum_{i=1}^L (\Delta C_{A_n} - n\mathbf{1} \otimes \mathbf{1})_{i,i+1} \\
&= \sum_{i=1}^L \left[(n+1) \sum_{\alpha=1}^{n(n+1)/2} ((e_\alpha)_i (f_\alpha)_{i+1} + (f_\alpha)_i (e_\alpha)_{i+1}) + \sum_{\alpha=1}^n \alpha(n+1-\alpha) (h_\alpha)_i (h_\alpha)_{i+1} \right. \\
&\quad \left. + \sum_{\alpha=1}^n \sum_{\beta=1}^{n-\alpha} \alpha(n+1-\alpha-\beta) ((h_\alpha)_i (h_{\alpha+\beta})_{i+1} + (h_{\alpha+\beta})_i (h_\alpha)_{i+1}) \right] - nL.
\end{aligned} \tag{86}$$

[Proof]. Set $h'' = (\Delta C_{A_n} - n\mathbf{1} \otimes \mathbf{1})$. Then

$$Q_{A_n} = \sum_{i=1}^L h''_{i,i+1}. \tag{87}$$

From (48) we observe that, for $\alpha \neq \beta$,

$$\begin{aligned}
h''_{\alpha \neq \beta} &= (n+1) [\delta_{\alpha, j(n+2)+k+2} \delta_{\beta, (j+1)(n+2)+k(n+1)} \\
&\quad + \delta_{\beta, j(n+2)+k+2} \delta_{\alpha, (j+1)(n+2)+k(n+1)}] \geq 0.
\end{aligned}$$

Therefore $(Q_{A_n})_{\alpha \neq \beta} \geq 0$, $\alpha, \beta = 1, 2, \dots, (n+1)^{L+1}$.

Again by (48) the sum of the elements in any given column β of the matrix h'' is

$$\begin{aligned}
\sum_{\alpha=1}^{(n+1)^2} h''_{\alpha\beta} &= \sum_{\alpha=1}^{(n+1)^2} (n+1) [\delta_{\alpha\beta} (\delta_{\alpha, l(n+1)+l+1} - 1) \\
&\quad + (n+1) [\delta_{\alpha, j(n+2)+k+2} \delta_{\beta, (j+1)(n+2)+k(n+1)} + \delta_{\beta, j(n+2)+k+2} \delta_{\alpha, (j+1)(n+2)+k(n+1)}] \\
&= \sum_{\alpha \neq l(n+1)+l+1}^{(n+1)^2} (n+1) [-\delta_{\alpha\beta} + \delta_{\alpha, j(n+2)+k+2} \delta_{\beta, (j+1)(n+2)+k(n+1)} \\
&\quad + \delta_{\beta, j(n+2)+k+2} \delta_{\alpha, (j+1)(n+2)+k(n+1)}] \\
&= (n+1) (-1 + \delta_{\beta, (j+1)(n+2)+k(n+1)} |_{\alpha=j(n+2)+k+2} + \delta_{\beta, j(n+2)+k+2} |_{\alpha=(j+1)(n+2)+k(n+1)}) \\
&= 0,
\end{aligned}$$

i.e., the sum of the elements in any given column β , $\beta = 1, 2, \dots, (n+1)^2$, of the matrix h'' is zero. Therefore the sum of the elements in any given column β , $\beta = 1, 2, \dots, (n+1)^{L+1}$, of the matrix Q_{A_n} is also zero, $\sum_{\alpha=1}^{(n+1)^{L+1}} (Q_{A_n})_{\alpha\beta} = 0$. At last $Q_{A_n} = H_{A_n} - (n+1)L$ is obviously A_n symmetric with the same spectrum (shifted by a constant) as H_{A_n} . ■

The long run distribution of the Markov chain described in Theorem 7 is given by the vector $\pi = (\pi_1, \pi_2, \dots)$, where π_i represents the “long time” probability of the state $i \in S$,

satisfying

$$\sum_{\alpha=1}^{(n+1)^{L+1}} (Q_{A_n})_{\alpha\beta} \pi_\alpha = 0, \quad \forall \beta \in S, \quad \sum_{\alpha=1}^{(n+1)^{L+1}} \pi_\alpha = 1. \quad (88)$$

However as this Markov chain is not irreducible, the solution of the equation (88) is not unique but depends on the initial conditions. From (48) we see that

$$(h'')_{\alpha\beta} = (h'')_{\beta\alpha} = 0, \quad \forall \beta, \\ \alpha = l(n+1) + l + 1, \quad l = 0, 1, \dots, n.$$

Hence from (87) we get

$$(Q_{A_n})_{\alpha\beta} = (Q_{A_n})_{\beta\alpha} = 0, \quad \alpha \in S_0, \quad \forall \beta,$$

with S_0 as in (83). Therefore if this Markov chain is initially at a given state $\alpha \in S_0$, it will remain at that state.

The states $\beta \notin S_0$ form a closed subset of S . From (48) and (87) one also learns that the absolute value of all the nonzero elements of any column of the intensity matrix Q_{A_n} are equal. Let S' be a closed subset of S with l elements. If the Markov chain is initially in the closed set S' , then it will remain in S' and the long run distribution is $\pi = (\pi_1, \pi_2, \dots, \pi_{(n+1)^{L+1}})$, where $\pi_i = 1/l$ for $i \in S'$ and $\pi_i = 0$ if $i \notin S'$.

5 Conclusion and Remark

Using the Casimir operators and coproduct operations of algebras, we have given a simple way to construct chain models with a certain algebraic symmetry and nearest or non-nearest neighbours interactions. We discussed integrable chain models with nearest neighbours interactions with the symmetries provided by the fundamental representation of the classical Lie algebra A_n . It is shown that corresponding to these A_n symmetric integrable chain models there are exactly solvable stationary discrete-time (resp. continuous-time) Markov chains whose spectra of the transition matrices (resp. intensity matrices) are the same as the ones of the corresponding integrable models.

Other symmetric integrable models (e.g. with B_n, C_n, D_n symmetry) and related TL algebraic structures and Markov chains can be investigated in a similar way. The discussion of integrable models related to higher dimensional representations of the algebras and

the integrability of chain models with non-nearest neighbours interactions is postponed to further work.

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